Automatic Tuning
of
PID-regulators

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Contents

Preface.

This text summarizes digital PID control and methods for automatic tuning based on relay feedback. The text is intended for use as a part of a graduate course in Adaptive Control and the topics are selected from the references with this in mind.

1 Introduction

Many process control problems can be adequately and routinely solved by conventional PID-control strategies. The overriding reason that the PID controller is so widely accepted is its simple structure which has proved to be very robust with regard to many commonly met process control problems as for instance disturbances and nonlinearities. Tuning of the PID settings is quite a subjective procedure, relying heavily on the knowledge and skill of the control engineer or even plant operator. Although tuning guidelines are available, the process of controller tuning can still be time consuming with the result that many plant control loops are often poorly tuned and full potential of the control system is not achieved. One way of deriving a self-tuning control law with the structure of a PID controller is to approximate the system with a parameterized model of order $n = 1$ or $n = 2$ and calculate the controller coefficients using a design method as e.g. pole placement, generalized predictive control or LQG. These methods are easily extended to a higher order with possibility of better control performance. They require a fair amount of a priori knowledge as for instance sampling period, dead time, model order, and desired response time. This knowledge may either be given by a skilled engineer or may be acquired automatically by some kind of experimentation. The latter alternative is preferable, not only to simplify the task of the operator, but also for the sake of robustness.

A key method for auto-tuning (Åström and Hägglund 1988) is to use relay feedback. Processes with the dynamics typically encountered in process control will then exhibit limit cycle oscillations and the auto tuner identifies one point on the Nyquist curve of the process from the experiment. When the operator decides to tune the controller, he simply presses a button starting a relay experiment. The relay feedback causes the process to oscillate with a controlled amplitude. The frequency of the limit cycle is approximately the ultimate frequency where the process has a phase lag of $180°$. The ratio of the amplitude of limit cycle and the relay amplitude is approximately the process gain at that frequency. Thus a point on the Nyquist curve of the open loop dynamics close to the ultimate point is determined. A reasonable PID controller based on this point may then be calculated.

2 Digital PID regulators

The ”textbook” version of the PID controller can be described by the equation

$$\begin{aligned} u(t) = \quad & K(y_r(t) - y(t)) + \frac{K}{T_i}\int_{-\infty}^{t}(y_r(\tau) - y(\tau))d\tau \\ & + KT_i\frac{d(y_r(t) - y(t))}{dt} \end{aligned} \tag{1}$$

where y_r is the set point and y is the process output. The PID controller was originally implemented using analog technology that went through several development stages.

In this development much know-how was accumulated imbedded into the analog design. Today, virtually all PID-regulators are implemented digitally. Early implementations were often a pure translation of equation 1 which left out many of the extra features that were incorporated in the analog design. The following implementation is discussed in detail in (Åström and Wittenmark, 1990).

Modification of ”textbook” version.

A pure derivative cannot and should not be implemented, because it will give a very large amplification of measurement noise. The gain of the derivative must thus be limited. This can be done by low-pass filtering the derivative term, resulting in a limited gain N at high frequencies. N is typically in the range of 3-20. In the work with analog controllers it was found advantageous not to let the derivative act on the command signal. It was also found suitable to let only a fraction b of the command signal act on the proportional part. Introducing the derivative operator $p = d/dt$ the PID-algorithm then becomes

$$\begin{aligned} u(t) = \quad & K(by_r(t) - y(t)) + \frac{K}{pT_i}(y_r(t) - y(t)) \\ & - \frac{pKT_i}{1 + pT_d/N} y(t) \end{aligned} \tag{2}$$

The different signal paths for the command signal and the process output in this equation gives different closed loop zeros which separate command signal response from the response from disturbances.

Discretization.

Any standard transformation method can be used in the discretization of equation (2). The following is a popular approximation that is easy to derive.

The proportional part

$$P(t) = K\{by_r(t) - y(t)\} \tag{3}$$

requires no transformation since it is purely static.

The integral part

$$I(t) = \frac{K}{T_i} \int^t (y_r(\tau) - y(\tau)) d\tau$$

The integration is approximated by

$$\begin{aligned} \int^{t+h} (y_r(\tau) - y(\tau)) d\tau \quad &= \quad \int^t (y_r(\tau) - y(\tau)) d\tau + \int_t^{t+h} (y_r(\tau) - y(\tau)) d\tau \\ &\approx \quad \int^t (y_r(\tau) - y(\tau)) d\tau \; + \; h(y_r(t) - y(t)) \end{aligned}$$

which gives the following (forward euler) time discretization:

$$I(t + h) = I(t) + \frac{Kh}{T_i}\{y_r(t) - y(t)\} \tag{4}$$

Table 1: RST-coefficients

$a_d = \frac{T_d}{T_d+Nh}$	$b_d = Na_d$	$b_i = \frac{h}{T_i}$
	$r_1 = -(1+a_d)$	$r_2 = a_d$
$s_0 = K(1+b_d)$	$s_1 = -K(1+a_d+2b_d-b_i)$	$s_2 = K(a_d+b_d-b_ia_d)$
$t_0 = Kb$	$t_1 = -K(b(1+a_d)-b_i)$	$t_2 = Ka_d(b-b_i)$

The derivative part

$$\frac{T_d}{N}\frac{dD(t)}{dt} + D(t) = -KT_d\frac{dy(t)}{dt}$$

is approximated by taking backward differences

$$\frac{T_d}{N}\frac{D(t)-D(t-h)}{h} + D(t) = -KT_d\frac{y(t)-y(t-h)}{h}$$

which give the following approximation:

$$D(t) = \frac{T_d}{T_d+Nh}D(t-h) - \frac{KT_dN}{T_d+Nh}\{y(t)-y(t-h)\} \tag{5}$$

If the delay operator q^{-1} defined as $q^{-1}x(t) \equiv x(t-h)$ is introduced, then combining equation (3),(4) and (5) gives the control signal

$$\begin{aligned} u(t) = K(by_r(t) - y(t) \quad & + \quad \frac{Kh}{T_i}\frac{q^{-1}}{1-q^{-1}}(y_r(t)-y(t)) \\ & - \quad \frac{KT_dN}{T_d+Nh}\,\frac{1-q^{-1}}{1-\frac{T_d}{T_d+Nh}q^{-1}}\;y(t) \end{aligned} \tag{6}$$

Equation (6) can after some calculations be written as

$$(1+r_1q^{-1}+r_2q^{-2})u(t) = (t_0+t_1q^{-1}+t_2q^{-2})y_r(t) - (s_0+s_1q^{-1}+s_2q^{-2})y(t) \tag{7}$$

with coefficients given by table 1.

Polynomial operators defined by:

$$\begin{aligned} R(q^{-1}) &= 1+r_1q^{-1}+r_2q^{-2} \\ S(q^{-1}) &= s_0+s_1q^{-1}+s_2q^{-2} \\ T(q^{-1}) &= t_0+t_1q^{-1}+t_2q^{-2} \end{aligned}$$

gives the following short form operator description of the PID regulator:

$$u(t) = \frac{1}{R(q^{-1})}\;\{T(q^{-1}y_r(t) - S(q^{-1})y(t)\} \tag{8}$$

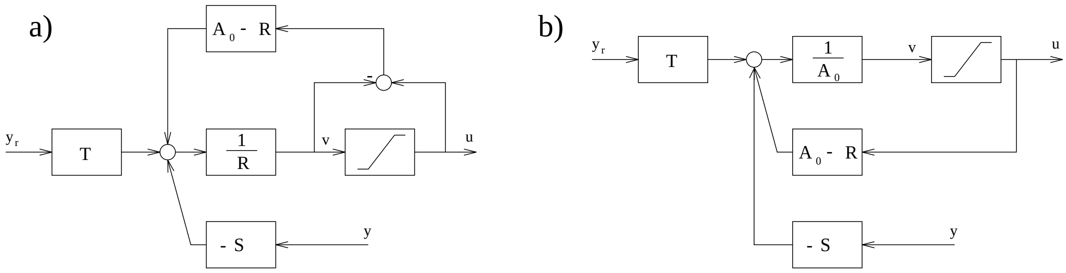

Figure 1: *Regulator with anti integrator windup and a) measured actuator output b) estimated actuator output*

Other realizations of $T(q^{-1})$ are often seen in the literature. Among the most common are

$$\begin{aligned} T(q^{-1}) &= s_0 + s_1 + s_2 \\ T(q^{-1}) &= S(q^{-1}) \end{aligned}$$

Integrator windup and bumpless transfer

If the control error $y_r(t) - y(t)$ is so large that the control output saturates the actuator, the feedback path will be "broken", because the actuator remains saturated even if the process output changes. The integrator may then integrate up to a very large value. When the error is finally reduced, the integral may be so large that it takes a considerable time to reach a normal value again. This effect is called integrator windup and one way to avoid it is shown in figure 1.a. An extra feedback path is provided in the controller by measuring the actuator output u and forming an error signal $u - v$, which is filtered and fed back through the integrator part of the controller $1/R(q^{-1})$. The error signal is zero when the actuator is not saturated and the block diagram gives equation (8). When the actuator saturates, the feedback signal will attempt to drive the error to zero, preventing the integrator from winding up. The filter A_0 determines how quickly the integral is reset. If the actuator output cannot be measured, the anti-windup scheme can be applied by incorporating a mathematical model of the saturating actuator. In this case the block diagram in figure 1.b gives the following equations:

$$\begin{aligned} v &= \frac{1}{A_0}\{(A_0 - R)u + Ty_r - Sy\} \\ u &= sat(v, u_{\min}, u_{\max}) \end{aligned}$$

If the regulator output in manual mode is given by $u(t) = u_{man}$ bumpless transfer from manual mode to controller mode is achieved by using

$$u_{\min} = u_{\max} = u_{man}$$

in manual mode, and then switching to the specified values for $u_{\min}$ and $u_{\max}$ in controller mode.

bumpless transfer from controller mode to manual mode is achieved by resetting u_{man} as follows:

$$u_{man} = u(t)$$

$$u_{\min} = u_{\max} = u_{man}$$

Example 1 *Influence of observer polynomial $A_o(q^{-1})$.*

Figure 2 shows the open loop response for the system:

$$\frac{2.8}{1+0.011s}$$

A PI-regulator with very short sampling time $T_s = 0.00005$ and the parameters $K = 25$, $T_i = 0.011$ and $b = 1$ is discretized by the transformations given in table 1 and the closed loop step response is shown in figure 3 with an observer polynomial $A_0 = 1$.

Figure 4 and figure 5 show the effect of a first order observer polynomial $A_0(q^{-1}) = 1 - \exp(-T_s/T_t)q^{-1}$ with time constant $T_t = 0.1T_i$ and $T_t = T_i$. Comparing figure 3 and figure 5 shows the importance of the observer polynomial for fast sampled systems.

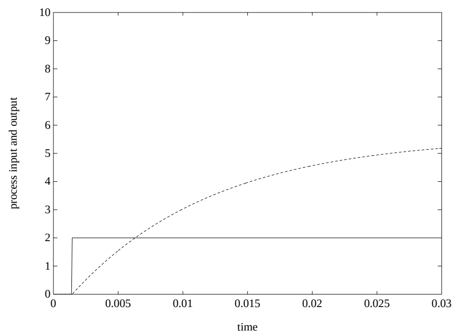

Figure 2: *Open loop response*

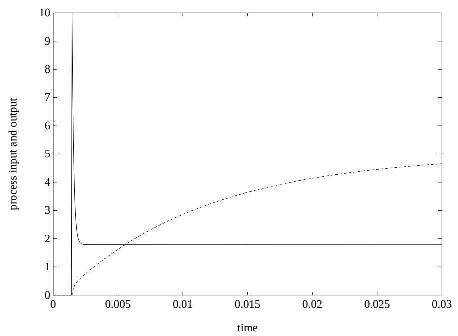

Figure 3: *Closed loop response with $A_o(q^{-1}) = 1$*

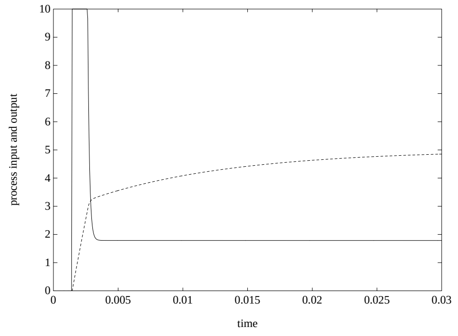

Figure 4: *Closed loop response with* $A_o(q^{-1}) = 1 - \exp(-T_s/0.1T_i)q^{-1}$

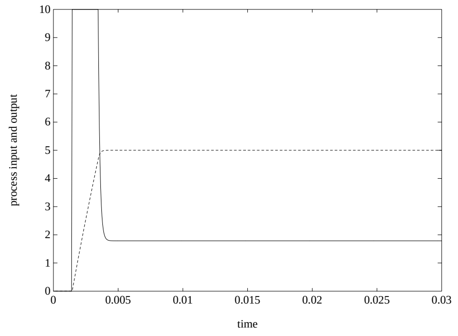

Figure 5: *Closed loop response with* $A_o(q^{-1}) = 1 - \exp(-T_s/T_i)q^{-1}$

3 Ziegler-Nichols tuning methods

The following parameters in the PID regulator have to be chosen:

K	proportional gain
T_i	integration time
T_d	derivative time
b	fraction of command signal
N	high frequency limiter of derivative action
$u_{\min}$	minimum saturation value
$u_{\max}$	maximum saturation value
h	sampling time

The primary parameters are K,T_i and T_d and the Ziegler-Nichols design methods for these parameters will be given. The parameter b only influence the zeros of the closed loop transfer function and will be set to 0.3 in the following. N can often be given a fixed default value, e.g., $N = 10$, which means that there is no derivative action of frequencies above N/T_d. The parameters $u_{\min}$ and $u_{\max}$ should be chosen close to the true saturation limits. The sampling period h must be chosen so short that the phase lead is not adversely affected by the sampling. This implies that the sampling period should be chosen so that $\frac{1}{h} \gg \frac{N}{T_d}$. This gives the following rule of thumb for regulators with derivative action

$$\frac{hN}{T_d} \approx 0.2 - 0.6$$

For PI regulators the rule of thumb is

$$\frac{h}{T_i} \approx 0.1 - 0.3$$

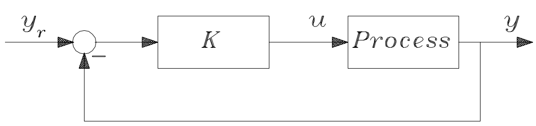

Figure 6: *Ziegler-Nichols ultimate period method*

In the ultimate sensitivity method the key idea is to determine the point where the Nyquist curve of the open loop system intersects the negative real axis. This is done by increasing the gain of a proportional controller in figure 6 until the closed loop system reaches the stability limit. The gain K_u and the corresponding period T_u of the oscillation are then determined and the PID coefficients are then found from Table 2.

In the step response method the unit step response of the process is determined experimentally (figure 3). It will be assumed that the step response is monotone except for a small initial part which eventually can have a non minimum phase characteristic. The tangent with the steepest slope and its intersection with the time axis are determined and the PID parameters are then obtained from table 3.

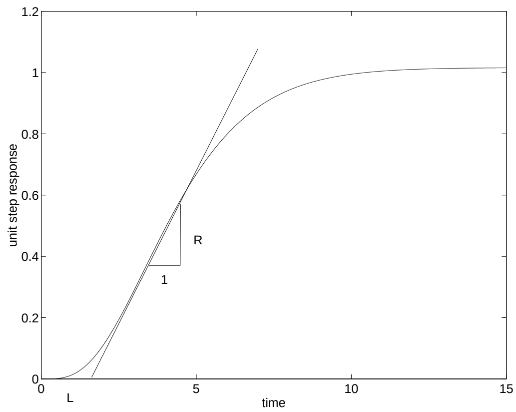

Figure 7: *Unit step response*

Table 2: *Ziegler-Nichols ultimate period method*

Regulator	K	T_i	T_d
P	$0.5K_u$		
PI	$0.45K_u$	$T_u/1.2$	
PID	$0.6K_u$	$T_u/2$	$T_u/8$

Table 3: *Ziegler-Nichols step response method*

Regulator	K	T_i	T_d
P	$1/RL$		
PI	$0.9/RL$	$3L$	
PID	$1.2/RL$	$2L$	$0.5L$

4 Auto Tuning based on relay feedback

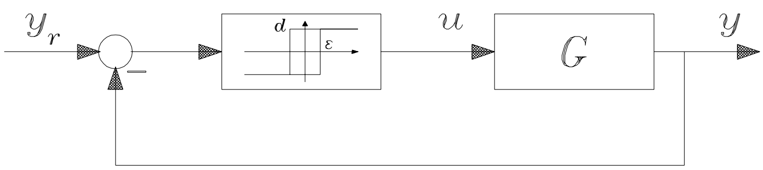

Figure 8: *Relay feedback*

A difficulty with the ultimate gain method, seen from an automated point of view, is that the closed loop system is at the stability boundary. A special method where an appropriate frequency of the input signal is generated automatically is achieved by introducing a nonlinear feedback of the relay type in the control loop. Consider the system figure 8. Assume that there is a limit cycle with period T_u so that the relay output is a periodically symmetric square wave. If the relay output is d, a simple Fourier series expansion of the relay output for $\epsilon = 0$ (no hysteresis) gives a first harmonic with amplitude $4d/\pi$. If it is further assumed that the process dynamic has low-pass character and that the contribution from the first harmonic dominates the output, then the error signal has the amplitude

$$a = \frac{4d}{\pi}\left|G(j\frac{2\pi}{T_u})\right|$$

The condition for oscillation is thus that

$$\begin{aligned} \arg G(j\tfrac{2\pi}{T_u}) &= -\pi \\ \left|G(j\tfrac{2\pi}{T_u})\right| &= \frac{\pi a}{4d} \end{aligned}$$

Defining

$$K_u = \frac{4d}{\pi a}$$

it is easily seen that with the above assumptions K_u is the gain that brings the system to stability boundary under pure proportional control. The ultimate gain K_u and the ultimate period T_u are thus easily found from a relay experiment.

Figure 9 shows a relay experiment giving the ultimate period T_u and gain K_u. The PID controller can then be derived from table 2 and a closed loop step response with this controller is shown on the figure too. This method has only one parameter that must be specified in advance, namely the amplitude of the relay. A feedback loop from measurement of amplitude of the oscillation to the relay amplitude can be used to ensure that the output is within reasonable bounds during oscillation. Even the order of magnitude of the time constant of the process can be unknown. Therefore, this method is not only suitable as a tuning device, it can also be used in a pre-tuning phase in other

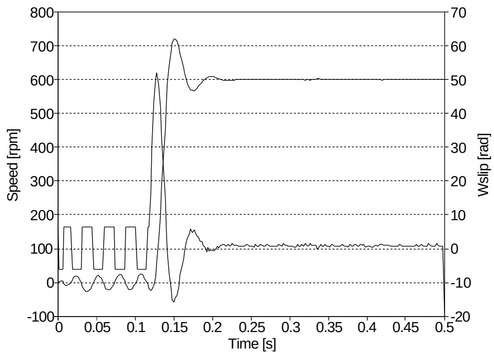

Figure 9: *Ziegler-Nichols tuning of a PID controller by a relay experiment*

tuning procedures where the time constant of the system has to be known, or it can be used to decide a suitable sampling period. Doing experiments with different amplitudes and hysteresis of the relay, several points on the Nyquist curve can be identified. Design methods based on two points can then be used.

Since limit cycling under relay feedback is a key idea of relay auto tuning, it is important to have methods for determination of the period and the amplitude of the oscillations.

Theorem 1 (Limit cycle period) *Assume that the system defined in figure 8 has a symmetric limit cycle with period T. This period is then the smallest value greater than zero satisfying the equation*

$$H_{T/2}(-1) = 0 \tag{9}$$

where $H_{T/2}(z)$ *is the zero order hold sampling of the continuous system* $G(s)$.

Proof: Let t_k denote the time where the relay switches to $u(t_k) = d$. Since it is assumed that the limit cycle is symmetric it follows that

$$t_k - t_{k-1} = \frac{T}{2}$$

and that the control signal has been constant $u(t) = -d$ for $t_{k-1} \leq t < t_k$. A time discrete control signal $u(t_{k-1}) = -d$ put on a DAC + zero-order hold circuit is seen to give the same analog $u(t)$. Hence

$$y(t_k) = H_{T/2}(q^{-1})\ u(t_k)$$

Since $q^{-i}u(t_k) = u(t_{k-i}) = (-d)^i$ and switching occurs when $y(t_k) = 0$ it follows that

$$0 = H_{T/2}(-1)\ d$$

which gives the equation (9).

Example 2 *Limit cycle period*

If the transfer function

$$G(s) = \frac{e^{-T_0 s}}{1 + s\tau}$$

is sampled with a sampling period h greater than T_0 the zero order hold sampling is given by

$$H_h(z) = \frac{b_1 z + b_2}{z(z + a1)} \tag{10}$$

with

$$b_1 = 1 - e^{-\frac{h - T_0}{\tau}} \quad b_2 = -e^{-\frac{h}{\tau}} + e^{-\frac{h - T_0}{\tau}} \quad a_1 = -e^{-h/\tau}$$

Hence the period $T/2$ is given by

$$H_{T/2}(-1) = \frac{-b_1 + b_2}{-1(-1 + a1)} = 0 \tag{11}$$

which gives

$$T = 2\tau \ln(2e^{\frac{T_0}{\tau}} - 1) \tag{12}$$

This period has to be compared with the ultimate period T_u defined by

$$\arg G(j\frac{2\pi}{T_u}) = -\frac{2\pi T_0}{T_u} - \tan^{-1}\frac{2\pi\tau}{T_u} = -\pi$$

For $\tau = 10$ and $T_0 = 3$ the equation gives $T_u = 1.02\,T$, where T is computed from equation (12) giving the period obtained from an relay experiment.

5 Tuning with specified phase and amplitude margin

Consider a situation where one point A on the Nyquist curve for the open loop system is known. With PID control it is possible to move the given point on the Nyquist curve to an arbitrary position in the complex plane, as shown in figure 10. The point may be moved in the direction of $G(j\omega)$ by changing the gain and in the orthogonal directions by changing the integral or the derivative time constants.

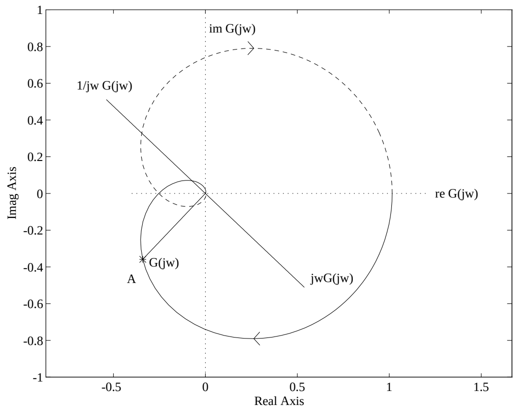

Figure 10: *Movement of the point A with a PID-regulator*

Consider a process with transfer function $G(s)$ as an example demonstrating the idea. The loop transfer function with PID control is

$$K\left\{1 + j\omega T_d + \frac{1}{j\omega T_i}\right\}G(j\omega)$$

The point where the Nyquist curve of $G(j\omega)$ intersects the negative real axis is given by the ultimate frequency $\omega_u = 2\pi/T_u$ and ultimate gain K_u, e.g. $K_u G(j\omega_u) = -1$. If

$$\omega_u T_d - \frac{1}{\omega_u T_i} = \tan\varphi_m \qquad (13)$$

the argument of the open loop transfer function is now $-\pi + \varphi_m$. If the magnitude of the open loop transfer function is specified to k_m simple trigonometric calculations give

$$K = k_m \frac{\cos\varphi_m}{|G(j\omega_u)|} = k_m K_u \cos\varphi_m$$

Equation 13 has many solutions for T_d and T_i. If however

Table 4: PID controller

Regulator	K	T_i	T_d
Ziegler-Nichols ultimate period	$0.60K_u$	$0.50T_u$	$0.125T_u$
$k_m = 0.5$ and $\phi_m = 45$ deg	$0.35K_u$	$0.76T_u$	$0.192T_u$

$$T_i = \alpha T_d$$

with α specified the equation 13 gives the following solution

$$T_d = \frac{1}{2\omega_u}\left\{\tan\varphi_m + \sqrt{4/\alpha + \tan^2\varphi_m}\right\}$$

Example 3 *Combined amplitude and phase margin specification*

A design method based on the requirement that the Nyquist curve intersects the circle with radius $k_m = \frac{1}{2}$ and an angle of $\varphi_m - 180° = -135°$ is given.

Specifying $\alpha = 4$ gives the following PID controller

$$
\begin{aligned}
T_d &= \tfrac{1+\sqrt{2}}{4\pi} T_u \\
T_i &= 4T_d \\
K &= \tfrac{\sqrt{2}}{4} K_u
\end{aligned}
$$

Table 4 compares the coefficients found by this method to the coefficients found by Ziegler-Nichols ultimate period method. K_u and T_u may be determined by relay feedback.

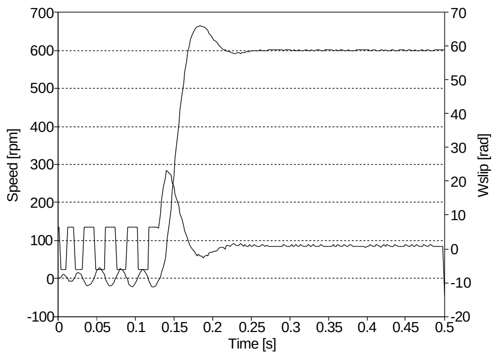

Figure 11: *Modified Ziegler-Nichols tuning of a PID-regulator by a relay experiment*

Figure 11 shows a relay experiment giving the ultimate period and gain. The PID controller can then be derived from table 4 and a closed loop step response with this controller is shown on the figure too.

6 Relay with hysteresis

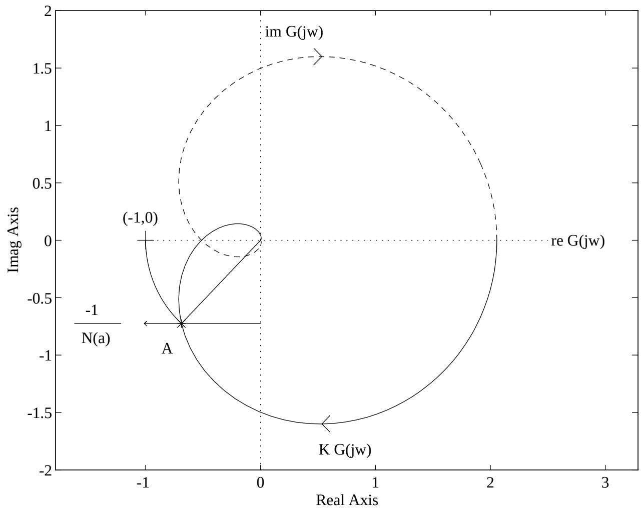

Figure 12: *Stable limit cycle oscillation*

With an ordinary relay, a small amount of noise can make the relay switch resulting in difficulties with determination of the period for a stable limit cycle. This can be overcome by introducing a relay with hysteresis as shown in figure 8. The describing function of such a relay is

$$-\frac{1}{N(a)} = -\frac{\phi}{4d}\sqrt{a_2 - \epsilon^2} - j\frac{\phi\epsilon}{4d} \tag{14}$$

where d is the relay amplitude and ϵ is the hysteresis width. This function is a straight line parallel to the real axis, in the complex plane. The intersection with the Nyquist curve A (figure 12) gives the stable limit cycle period and amplitude. By choosing the relation between ϵ and d it is possible to specify the point A with a given imaginary part.

This property can be used to obtain a regulator witch gives the system a desired phase margin. Consider a process $G(s)$ controlled by a proportional regulator. The loop transfer function is thus $KG(s)$. If the specified phase margin is ϕ_m the relay characteristics has to be chosen so the describing function goes through the point P. From figure 12 and equation 14 the following equations are obtained

$$\begin{aligned} d &= \frac{\pi a}{4} \\ \epsilon &= a \sin \phi_m \end{aligned} \tag{15}$$

where a is the amplitude of process output oscillation.

Integral and derivative action can then be included, using the method described in the previous section.

7 Offset

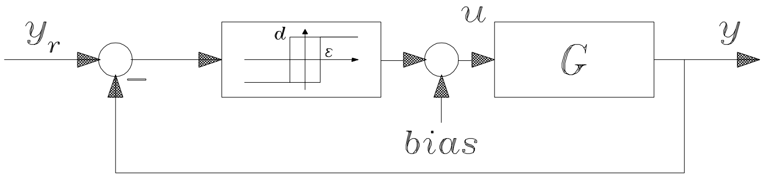

Figure 13: *Relay feedback with corrective bias*

In a typical auto-tuning setup (figure 13) a supervisory program will be activated to initialize the control error to zero and the bias to an appropriate level. Furthermore any request for set point changes is disabled during tuning to ensure that the relay will receive a zero mean input signal. Any static load disturbance or incorrect bias initialization will cause asymmetry in the relay switching intervals and thus give incorrect estimates of the ultimate gain and period.

If t_1 and t_2 are the intervals of positive and negative relay outputs respectively, and d is the relay amplitude, then the corrective bias may be computed for each positive going shift of the relay by:

$$\Delta bias = d\,\frac{t_1 - t_2}{t_1 + t_2}$$

Hence the supervisory program can monitor the switching intervals and automatically add the bias correction once the asymmetry condition is detected.

8 Auto Tuning based on Pole Placement

Suppose that the process to be controlled is characterized by the model

$$y(t) = \frac{B^*(q^{-1})}{A^*(q^{-1})}u(t) + v(t)$$

then the controller Eq. 8 gives the closed loop response

$$\begin{aligned} y(t) \quad &= \quad \frac{B^*(q^{-1})T^*(q^{-1})}{A^*(q^{-1})R^*(q^{-1}) + B^*(q^{-1})S^*(q^{-1})}y_r(t) \\ &\quad + \frac{A^*(q^{-1})R^*(q^{-1})}{A^*(q^{-1})R^*(q^{-1}) + B^*(q^{-1})S^*(q^{-1})}v(t) \end{aligned} \tag{16}$$

with characteristic equation

$$A(z)R(z) + B(z)S(z) = P(z) \tag{17}$$

where the relation between $X(z)$ and $X^*(z^{-1})$ for a n'th order polynomial is given by

$$X(z) = z^n X^*(z^{-1})$$

If we specify a stable closed loop polynomial $P(z) = A_m(z)A_o(z)$ and require that $R(z=1) = 0$ (integral action) then equation (17) has a unique solution for $R(z)$ and $S(z)$ if $A(z)$ and $B(z)$ have no common roots and

$$\deg R = \deg S = \deg A_m = \deg A_o = \deg A$$

The polynomial $T^*(q^{-1})$ may be computed in different ways e.g.:

$$\begin{aligned} T^*(q^{-1}) &= t_0 A_o^*(q^{-1}) \\ T^*(q^{-1}) &= s_0 + s_1 + ... + s_n \\ T^*(q^{-1}) &= S^*(q^{-1}) \end{aligned}$$

For $T^*(q^{-1}) = t_0 A_o^*(q^{-1})$ equation (16) gives:

$$y = \frac{t_0 B}{A_m} y_r + \frac{AR}{A_o A_m} v$$

DC-gain equal to one is achieved for

$$t_0 = \frac{s_0 + s_1 + ... + s_n}{1 + a_{o1} + ... + a_{on}} = \frac{S(z=1)}{A_o(z=1)}$$

If the ultimate gain k_u,ultimate period T_u and the delay time T_{del} are determined by a relay experiment in a pre-tuning phase, then suitable values for the sampling period and the closed loop dynamics may be chosen.

Example 4 *Second order model*

1. A relay experiment gives the ultimate period T_u and the delay T_{del}
2. The sampling time is chosen as $T_s = T_{del}$
3. Based on the relay experiment the closed loop poles $P(z) = z^4 + p_1 z^3 + ... + p_4 = A_0(z)A_m(z)$ may for instance be specified as

   $$\begin{aligned} A_o(z) &= \{z + a_o\}^2 \\ A_m(z) &= z^2 + a_{m1} z + a_{m2} \end{aligned}$$

   with coefficients given by

   $$\begin{aligned} a_o &= -\exp\left\{-\beta \tfrac{T_s}{T_i}\right\} = -\exp\left\{-\beta \tfrac{2T_s}{T_u}\right\} \\ a_{m1} &= -2\exp\left\{-\zeta\alpha \tfrac{2\pi}{T_u} T_s\right\} \cos(\alpha \tfrac{2\pi}{T_u} T_s \sqrt{1-\zeta^2}) \\ a_{m2} &= \exp\left\{-2\zeta \tfrac{2\pi}{T_u} T_s\right\} \end{aligned}$$

   and tuning parameters chosen as

   $$\begin{aligned} \alpha &= 0.5 \\ \beta &= 1 \\ \zeta &= \tfrac{\sqrt{3}}{2} \end{aligned}$$

4. A least square estimation method gives the model:

$$\begin{aligned} A(z) &= z^2 + a_1 z + a_2 \\ B(z) &= b_1 z + b_2 \end{aligned}$$

5. The controller coefficients:

$$\begin{aligned} R(z) &= z^2 + r_1 z + r_2 \\ S(z) &= s_0 z^2 + s_1 z + s_2 \\ T(z) &= \tfrac{s_0+s_1+s_2}{1+2a_0+a_0^2} A_0(z) \end{aligned}$$

in the controller

$$u(t) = \frac{T(q)}{R(q)} y_r(t) - \frac{S(q)}{R(q)} y(t)$$

are calculated by solving the equations:

$$A(z)R(z) + B(z)S(z) = A_o(z)A_m(z)$$

$$R(z = 1) = 0$$

The solution may be written in the following matrix form:

$$\begin{bmatrix} b_1 & 0 & 0 & 1 & 0 \\ b_2 & b_1 & 0 & a_1 & 1 \\ 0 & b_2 & b_1 & a_2 & a_1 \\ 0 & 0 & b_2 & 0 & a_2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} p_1 - a_1 \\ p_2 - a_1 \\ p_3 \\ p_4 \\ -1 \end{bmatrix}$$

References

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[2] Åström K.J. and B. Wittenmark. Computer-Controlled Systems. Prentice-Hall (ISBN 0-13-172784-2), 1990.